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Solution of State-Constrained Optimal Control Problems Through Quasilinearization

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SUMMARY
Quasilinearization is extended to the numerical solution of multi-point boundary-value problems for ordinary differential equations. It is described how such problems arise from state-constrained optimal control problems. A simple numerical example is given to illustrate the method.

1. Introduction

Quasilinearization [1–4] is essentially a Newton–Raphson method for the numerical solution of boundary-value problems for differential equations. It has turned out to be an effective tool, in particular in the solution of the two-point boundary-value problems that result from the application of Pontryagin's maximum principle to problems of optimal control [5,6]. Long [6] has provided an important contribution to the method by extending it to problems with unknown terminal time.

It is shown in this paper that quasilinearization is easily applied to multi-point boundary-value problems, such as arise in optimal control problems with state constraints [7–10]. Long's method [6] is used to remove the difficulty that the times at which the intermediate boundary conditions apply are not known.

In Section 2 of this paper a general multi-point boundary-value problem for ordinary differential equations is formulated. In Section 3 the application of quasilinearization to the numerical solution of such problems is outlined, while in Section 4 it is reviewed how state-constrained optimal control problems give rise to multi-point boundary-value problems. Finally in Section 5 the method is illustrated with a simple numerical example.

2. Formulation of Multi-Point Boundary-Value Problems

Before introducing a general formulation of multi-point boundary-value problems the following notation is adopted. Let n be a natural number. Then

$$a_1, a_2, \ldots, a_\alpha \tag{1}$$

represents a sequence of $\alpha \leqq n$ different natural numbers all less than or equal to n, arranged in order of increasing magnitude. If the numbers in the sequence and the length of the sequence depend upon a parameter j, the sequence is denoted as

$$a_1^j, a_2^j, \ldots, a_{\alpha^j}^j . \tag{2}$$

For a given sequence of this form, the sequence

$$\bar{a}_1^j, \bar{a}_2^j, \ldots, \bar{a}_{\bar{\alpha}^j}^j \tag{3}$$

is defined as the sequence of $\bar{\alpha}^j = n - \alpha^j$ natural numbers less than or equal to n that are *not* in the sequence (2). Also (3) is arranged in order of increasing magnitude.

A general multi-point boundary-value problem may now be formulated as follows. Consider $m+1$ values $t_0 < t_1 < t_2 \ldots < t_m$ of the independent variable t, which will be referred to as time. These instants define m intervals on the time axis. Suppose that on each interval a system of n first-order autonomous simultaneous differential equations is given. For the j-th interval $[t_{j-1}, t_j]$ this system is represented in vector form as

$$\dot{\boldsymbol{y}}(t) = \boldsymbol{f}^j(\boldsymbol{y}(t)), \qquad j = 1, 2, \ldots, m \tag{4}$$

where $\boldsymbol{y}$ is an n-vector and $\boldsymbol{f}$ an n-dimensional vector function. For the j-th interval, $j = 1, 2, \ldots, m$, the following *explicit boundary conditions* are given:

$$\begin{aligned} y_i(t_{j-1}^+) &= {}^0y_i^j, \qquad i = a_1^j, a_2^j, \ldots, a_{\alpha^j}^j \\ y_i(t_j^-) &= {}^1y_i^j, \qquad i = b_1^j, b_2^j, \ldots, b_{\beta^j}^j \end{aligned} \tag{5}$$

where $\alpha^j \leqq n$ and $\beta^j \leqq n$. The $y_i(t)$, $i = 1, 2, \ldots, n$, are the components of $\boldsymbol{y}(t)$.

For the transition from the $(j-1)$-th interval to the j-th interval, $j = 2, 3, \ldots, m$, the following *continuity conditions* hold:

$$y_i(t_{j-1}^-) = y_i(t_{j-1}^+), \qquad i = c_1^j, c_2^j, \ldots, c_{\gamma^j}^j \tag{6}$$

where $\gamma^j \leqq n$.

Moreover, for $j = 1, 2, \ldots, m-1$, the following *implicit boundary conditions* may be given:

$$g_k^j(\boldsymbol{y}(t_j^-), \boldsymbol{y}(t_j^+)) = 0, \qquad k = d_1^j, d_2^j, \ldots, d_{\delta^j}^j \tag{7}$$

where $\delta^j \leqq n$, and where the g_k are given functions. Also at the initial and terminal times there may be implicit boundary conditions of the form

$$\begin{aligned} g_k^0(\boldsymbol{y}(t_0^+)) &= 0, \qquad k = d_1^0, d_2^0, \ldots, d_{\delta^0}^0 \\ g_k^m(\boldsymbol{y}(t_m^-)) &= 0, \qquad k = d_1^m, d_2^m, \ldots, d_{\delta^m}^m \end{aligned} \tag{8}$$

where $\delta^0 \leqq n$ and $\delta^m \leqq n$.

The problem is now to find a solution to the differential equations (4), satisfying the explicit boundary conditions (5), the continuity conditions (6) and the implicit boundary conditions (7) and (8). To solve the differential equations $m \cdot n$ initial conditions (n for each of the m intervals) are required. If the intermediate times $t_1, t_2, \ldots, t_{m-1}$ and the terminal time t_m are unknown, another m data are needed. The total number of equations that is available for finding these $(n+1)m$ unknowns is

$$\sum_{j=1}^{m} (\alpha^j + \beta^j) + \sum_{k=2}^{m} \gamma^k + \sum_{j=0}^{m} \delta^j . \tag{9}$$

This number must therefore equal $(n+1)m$.

3. Solution of Multi-Point Boundary-Value Problems Through Quasilinearization

In this Section the method of quasilinearization is developed for the multi-point boundary-value problem of the preceding Section. To eliminate the difficulty that the intermediate times $t_1, t, \ldots, t_{m-1}$ and the terminal time t_m are unknown, on each interval $[t_{j-1}, t_j]$, $j = 1, 2, \ldots, m$ a dummy time variable τ is introduced by the substitution

$$t = t_{j-1} + \vartheta_j \tau . \tag{10}$$

Here

$$\vartheta_j = t_j - t_{j-1} \tag{11}$$

is the unknown duration of the interval. This substitution transforms the differential equation (4) into

$$\dot{\boldsymbol{y}}^j(\tau) = \vartheta_j \boldsymbol{f}^j(\boldsymbol{y}^j(\tau)), \qquad j = 1, 2, \ldots, m, \ \tau \in [0, 1] \tag{12}$$

where

$$\boldsymbol{y}^j(\tau) = \boldsymbol{y}(t_j + \vartheta_j \tau) \tag{13}$$

denotes the solution on the j-th interval. Now on the j-th interval the unknown duration ϑ_j is adjoined as a $(n+1)$-th component y^j_{n+1} of the dependent variable $\boldsymbol{y}^j$, which satisfies the differential equation

$$\dot{y}^j_{n+1}(\tau) = 0. \tag{14}$$

The vector function $\boldsymbol{y}^j(\tau)$ and the additional component $y^j_{n+1}(\tau)$ may be combined into a single vector that for convenience is denoted by the same symbol $\boldsymbol{y}^j(\tau)$. By combining (12) and (14) it is seen that this augmented vector satisfies a differential equation of the form

$$\dot{\boldsymbol{y}}^j(\tau) = \boldsymbol{f}^j(\boldsymbol{y}^j(\tau)), \qquad j = 1, 2, \ldots, m, \ \tau \in [0, 1], \tag{15}$$

where for convenience also the same symbol $\boldsymbol{f}$ is employed.

The explicit boundary conditions for each interval may now be written as

$$\begin{aligned} y^j_i(0) &= {}^0y^j_i, \qquad i = a^j_1, a^j_2, \ldots, a^j_{\alpha j} \\ y^j_i(1) &= {}^1y^j_i, \qquad i = b^j_1, b^j_2, \ldots, b^j_{\beta j} \end{aligned} \tag{16}$$

for $j = 1, 2, \ldots, m$. The continuity conditions may be represented as

$$y^{j-1}_i(1) = y^j_i(0), \qquad i = c^j_1, c^j_2, \ldots, c^j_{\gamma j} \tag{17}$$

for $j = 2, 3, \ldots, m$. The implicit boundary conditions may be converted to

$$\begin{aligned} &g^0_k(\boldsymbol{y}^1(0)) = 0, && k = d^0_1, d^0_2, \ldots, d^0_{\delta 0} \\ &g^j_k(\boldsymbol{y}^j(1), \boldsymbol{y}^{j+1}(0)) = 0, && k = d^j_1, d^j_2, \ldots, d^j_{\delta j} \\ &g^m_k(\boldsymbol{y}^m(1)) = 0, && k = d^m_1, d^m_2, \ldots, d^m_{\delta m} \end{aligned} \tag{18}$$

where $j = 1, 2, \ldots, m-1$.

Quasilinearization consists of replacing the nonlinear differential equation (15) with a linearized version. The method is iterative. Suppose that after the k-th iteration an approximate solution ${}_k\boldsymbol{y}^j$ has been obtained, and suppose that $\boldsymbol{y}^j$ is the actual solution, satisfying all boundary conditions. Then if the approximation ${}_k\boldsymbol{y}^j$ and the actual solution $\boldsymbol{y}^j$ are not very different, the differential equation (15) may be approximated by

$$\dot{\boldsymbol{y}}^j(\tau) \approx \boldsymbol{f}^j({}_k\boldsymbol{y}^j(\tau)) + \boldsymbol{J}^j({}_k\boldsymbol{y}^j(\tau))[\boldsymbol{y}^j(\tau) - {}_k\boldsymbol{y}^j(\tau)] \tag{19}$$

where $\boldsymbol{J}^j(\boldsymbol{y})$ denotes the Jacobian of $\boldsymbol{f}^j$, i.e., $\boldsymbol{J}^j(\boldsymbol{y})$ is a matrix of which the (k, l)-th entry is given by

$$J^j_{kl}(\boldsymbol{y}) = \frac{\partial f^j_k(\boldsymbol{y})}{\partial y_l}.$$

The new approximation ${}_{k+1}\boldsymbol{y}^j$ is now obtained by replacing the approximately equal sign with an equal sign:

$${}_{k+1}\dot{\boldsymbol{y}}^j(\tau) = \boldsymbol{J}^j({}_k\boldsymbol{y}^j(\tau))[{}_{k+1}\boldsymbol{y}^j(\tau) - {}_k\boldsymbol{y}^j(\tau)] + \boldsymbol{f}^j({}_k\boldsymbol{y}^j(\tau)). \tag{20}$$

This is a linear differential equation of the form

$${}_{k+1}\dot{\boldsymbol{y}}^j(\tau) = {}_k\boldsymbol{A}^j(\tau)\,{}_{k+1}\boldsymbol{y}^j(\tau) + {}_k\boldsymbol{b}^j_k(\tau), \qquad j = 1, 2, \ldots, m \tag{21}$$

where the matrix ${}_k\boldsymbol{A}^j(\tau)$ and the vector ${}_k\boldsymbol{b}^j(\tau)$ depend upon the trajectory that was found at the k-th iteration. These linear differential equations are solved for ${}_{k+1}\boldsymbol{y}^j$, $j = 1, 2, \ldots, m$, while satisfying the boundary and continuity conditions exactly. This may be done as follows:

First generate by numerical integration $\bar{\alpha}^1$ solutions of the homogeneous equation for the first interval

$$\boldsymbol{y}^1(\tau) = {}_k\boldsymbol{A}^1(\tau)\boldsymbol{y}^1(\tau) \tag{22}$$

with the initial conditions

$$y^1(0) = \mathrm{col}(0, 0, \ldots, 1, 0, \ldots, 0) \tag{23}$$

where the 1 is in the $\bar{a}_i^1$-th position, for $i=1, 2, \ldots, \bar{\alpha}^1$. Denote these solutions as $z_1^{11}, z_2^{11}, \ldots, z_{\bar{\alpha}^1}^{11}$. Next consider the homogeneous equation in the second interval

$$_{k+1}y^2(\tau) = {}_kA^2(\tau)_{k+1}y^2(\tau). \tag{24}$$

First generate $\bar{\alpha}^1$ solutions, denoted as $z_1^{21}, z_2^{21}, \ldots, z_{\bar{\alpha}^1}^{22}$ with the initial conditions

$$z_i^{21}(0) = z_i^{11}(0), \qquad i=1, 2, \ldots, \bar{\alpha}^1. \tag{25}$$

Some of the components of the solution may have a discontinuity. Therefore, generate another $\bar{\gamma}^2$ solutions $z_1^{22}, z_2^{22}, \ldots, z_{\bar{\gamma}^2}^{22}$ of the homogeneous equation with the initial conditions

$$z_i^{22}(0) = \operatorname{col}(0, 0, \ldots, 0, 1, 0, \ldots, 0) \tag{26}$$

where the 1 is in the $\bar{c}_i^2$-th position, for $i=1, 2, \ldots, \bar{\gamma}^2$. Thus during the second interval a total of $\bar{\alpha}^1 + \bar{\gamma}^2$ solutions of the homogeneous equations have been obtained.

During the third interval first two sets of homogeneous solutions are generated, namely $z_1^{31}, z_2^{31}, \ldots, z_{\bar{\alpha}^1}^{31}$ and $z_1^{32}, z_2^{32}, \ldots, z_{\bar{\gamma}^2}^{32}$, which satisfy the initial conditions

$$\begin{aligned} z_i^{31}(0) &= z_i^{21}(1), \qquad i = 1, 2, \ldots, \bar{\alpha}^1 \\ z_i^{31}(0) &= z_i^{22}(1), \qquad i = 1, 2, \ldots, \bar{\gamma}^2 \end{aligned} \tag{27}$$

Furthermore a third set of homogeneous solutions $z_1^{33}, z_2^{33}, \ldots, z_{\bar{\gamma}^3}^{33}$ is computed, which satisfy the initial conditions

$$z_i^{33}(0) = \operatorname{col}(0, 0, \ldots, 0, 1, 0, \ldots, 0) \tag{28}$$

where the 1 is in the $\bar{c}_i^3$-th position, for $i=1, 2, \ldots, \bar{\gamma}^3$. Continuing in this manner during the j-th interval j sets of solutions of the homogeneous equations are obtained.

Finally a particular solution is found. In the first interval this particular solution z_p^1 is computed by numerically integrating the inhomogeneous equation (21) during the first interval with the initial conditions

$$z_{p,i}^1(0) = {}^0y_i^1 , \qquad i=a_1^1, a_2^1, \ldots, a_{\alpha j}^1 \tag{29}$$

where $z_{p,i}^1$ denotes the i-th component of z_p^1. The remaining components of $z_p^1(0)$ may be chosen arbitrarily. During the second interval a particular solution z_p^2 of the inhomogeneous equation is obtained with the initial condition

$$z_p^2(0) = z_p^1(1) \tag{30}$$

Continuing in this manner a particular solution is obtained for each interval.

The general solution during the j-th interval may now be represented as follows

$$\begin{aligned} {}_{k+1}y^j(\tau) = {} & \lambda_1^1 z_1^{j1}(\tau) + \lambda_2^1 z_2^{j1}(\tau) + \ldots + \lambda_{\bar{\alpha}^1}^1 z_{\bar{\alpha}^1}^{j1}(\tau) \\ & + \lambda_1^2 z_1^{j2}(\tau) + \lambda_2^2 z_2^{j2}(\tau) + \ldots + \lambda_{\bar{\gamma}^2}^2 z_{\bar{\gamma}^2}^{j2}(\tau) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \lambda_1^j z_1^{jj}(\tau) + \lambda_2^j z_2^{jj}(\tau) + \ldots + \lambda_{\bar{\gamma}^j}^j z_{\bar{\gamma}^j}^{jj}(\tau) \\ & + z_p^j(\tau). \end{aligned} \tag{31}$$

Here $\lambda_1^1, \ldots, \lambda_{\bar{\gamma}^j}^j$ are scalar constants to be determined. Each function $z^{ji}(\tau)$ represents the effect in the j-th interval of a perturbation in the initial condition at the beginning of the i-th interval.

The total number of constants for the last interval is

$$\bar{\alpha}^1 + \sum_{j=2}^{m} \bar{\gamma}^j \tag{32}$$

The number of boundary conditions that have not been used is

$$\sum_{j=2}^{m} \alpha^j + \sum_{j=1}^{m} \beta^j + \sum_{j=0}^{m} \delta^j \tag{33}$$

It is easily verified that the numbers (32) and (33) are equal if sufficient boundary conditions are given.

4. Solution of State-Constrained Optimal Control Problems Through Quasilinearization

In this Section state-constrained optimal control problems are briefly reviewed. Consider an autonomous differential system

$$\dot{x}(t) = f(x(t), u(t)) \tag{34}$$

It is desired to transfer the state $x(t)$ from an initial point $x(t_0) = x_0$ to a terminal point $x(t_f) = x_f$ such that

$$\int_{t_0}^{t_f} f_0(x(t), u(t)dt \tag{35}$$

is minimal. For the solution a set of inequalities of the form

$$g_i(x(t), u(t)) \leqq 0, \qquad i = 1, 2, \ldots, l \tag{36}$$

must be satisfied, where g_i may depend on either x or u, or both.

It is well-known [7–11] that the optimal trajectory is divided into a number of sections where each section is on a different combination of boundaries. In each section the solution is determined by a set of $2n$ simultaneous differential equations, n for the state and n for the adjoint variable. Suppose that the optimal trajectory is divided into m sections; then $m(2n+1)$ data are required to solve the multi-point boundary value problem. From the maximum principle of Pontryagin the following data may be derived [8]:

(a) The initial state is given; this yields n data.

(b) At the terminal time n boundary conditions for the state and adjoint variable are given; this yields another n data.

(c) In the case of undetermined terminal time the Hamiltonian is zero at the terminal time; if the terminal time is fixed the condition

$$\sum_{j=1}^{m} \vartheta_j = t_f - t_0 \tag{37}$$

must be satisfied; in either case one equation results.

(d) At each junction point the state is continuous; this gives $(m-1)n$ data.

(e) At each junction point the Hamiltonian is continuous; this provides $m-1$ equations.

(f) At each junction point each unknown discontinuity in a component of the adjoint variable is compensated by an additional condition of the form $h(x(t_j)) = 0$. This results in another $(m-1)n$ data.

Altogether precisely the required number of equations $(2n+1)m$ is obtained.

The multi-point boundary value problem which results in this manner may be solved by the technique outlined in Section 3. The method applies to state-constrained problems, but also to bang-bang type problems where quasilinearization is known sometimes to give difficulties [12].

5. Example

One of the examples that have been used to test the method is the following problem, which is taken from Niemann [12]. Consider the system described by the equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= [1 - x_1^2(t)]x_2(t) - x_1(t) + u(t) \end{aligned} \tag{38}$$

with the initial state

$$\begin{aligned} x_1(0) &= 1 \\ x_2(0) &= 0 \end{aligned} \tag{39}$$

The system is to be brought to the terminal state

$$\begin{aligned} x_1(5) &= -0.05 \\ x_2(5) &= -0.05 \end{aligned} \tag{40}$$

while minimizing the criterion

$$\int_0^5 [x_1^2(t)+x_2^2(t)+u^2(t)]\,dt. \tag{41}$$

The state is constrained as follows:

$$x_2(t) \geqq 0.4\,, \qquad 0 \leqq t \leqq 5. \tag{42}$$

For the *unconstrained* problem the Hamiltonian is

$$H = x_1^2+x_2^2+u^2-p_1x_2-p_2[(1-x_1^2)x_2-x_1+u] \tag{43}$$

where p_1 and p_2 are the components of the adjoint variable. It follows that the optimal input is given by

$$u^0 = \tfrac{1}{2}p_2 \tag{44}$$

while p_1 and p_2 satisfy the differential equations

$$\begin{aligned} -\dot{p}_1 &= 2x_1+2p_2x_1x_2+p_2 \\ -\dot{p}_2 &= 2x_2-p_1-p_2(1-x_1^2). \end{aligned} \tag{45}$$

Numerical solution of the two-point boundary-value problem defined by (38), (39), (40), (44) and (45) resulted in the trajectory of Fig. 1. Convergence was obtained in four iterations.

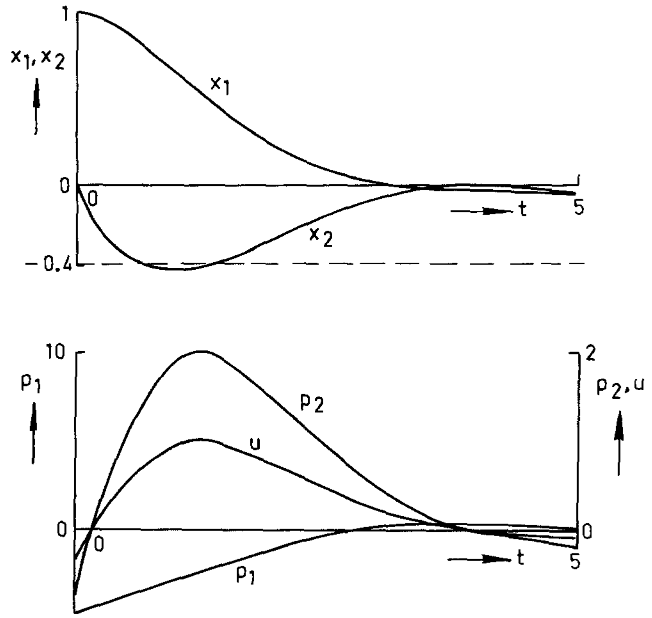

Figure 1. Solution of unconstrained problem.

It is observed that the constraint (42) is violated during part of the maneuver. It is reasonable to assume that the optimal trajectory consists of three parts: The first part off the boundary, the second part on the boundary, and the third part off the boundary again. It may be shown that while the system stays on the boundary the adjoint variables satisfy the differential equations

$$\begin{aligned} -\dot{p}_1 &= 2x_1+(2x_1x_2+1)[2x_1-2(1-x_1^2)x_2] \\ -\dot{p}_2 &= 2x_2-p_1-(1-x_1^2)[2x_1-2(1-x_1^2)x_2] \end{aligned} \tag{46}$$

while the optimal input follows from

$$u^0 = x_1 - (1 - x_1^2)x_2 . \tag{47}$$

On the boundary the Hamiltonian is given by

$$H = x_1^2 + x_2^2 + u^2 - p_1 x_2 . \tag{48}$$

Finally it may be derived that at the point where the trajectory enters the boundary p_1 is continuous, but p_2 jumps by an unknown amount. It is known that x_2 is 0.4 at the entry point, however, which provides the required equation. At the exit from the boundary both p_1 and p_2 are continuous. The times at which entry and exit occur are unknown; compensating equations are provided by the fact that the Hamiltonian is continuous both at the entry and the exit point.

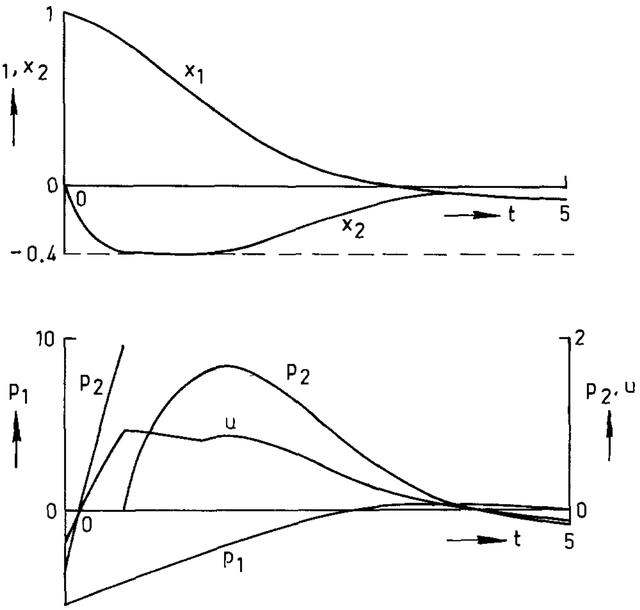

Figure 2. Solution of constrained problem.

The resulting four-point boundary-value problem was solved through quasilinearization. Also here convergence was obtained in four iterations. Fig. 2 shows the optimal trajectory. It is seen that the trajectory is not much affected by the introduction of the constraint. The minimum value of the criterion increases from 2.879 for the unconstrained case to 2.962 for the constrained case.

6. Conclusions

In this paper a method has been outlined to apply quasilinearization to multi-point boundary-value problems such as arise from state-constrained optimal control problems. The method is an efficient technique for dealing with such complicated problems. The method has been successfully applied not only to state-constrained control problems, but also to linear bang-bang control problems where single-interval quasilinearization fails to produce a solution [13].

In the relatively simple problems that were tackled during the course of the investigation convergence difficulties were seldom encountered. In general it is to be expected, however, that fast convergence and accurate solutions may be obtained provided a reasonably good initial estimate is available. The method has this in common with all Newton-Raphson techniques.

Further work will be directed along the following lines:

(1) Reduction of the dimension of the simultaneous differential equations when the optimal

trajectory is on a boundary [11]. This procedure at the same time removes the discontinuities at the entry points.

(2) Investigation into a suggestion of Niemann [12] that allows the automatic determination of the order in which the various sections of the optimal trajectory on and off a boundary occur.

(3) Investigation into a method to prevent the unknown durations of intervals ϑ_j from assuming negative values. This sometimes leads to spurious solutions.

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